Topology for Algebraic Geometry in Lean

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Closed sets are stable under specialization

Definition: x specializes to y if y is in the closure of {x}

Show: A closed set contains the specializations of all its points

Indiscrete topology example

```
lemma closed_spec_stable (C : Closeds X) (c : X) (hC: c ∈ C) : ∀ x : X, (spec c x) → x ∈ C := by
  intro x hX
  rw [spec] at hX
  have H : closure {c} ⊆ C.carrier := by
    rw [←IsClosed.mem_iff_closure_subset]
    exact hC
    exact C.closed'
  show x ∈ C.carrier
  exact H hX
```

- If c specializes to x, x is in the closure of c
- Since c is in C, the closure of c is in C
- Since x is in the closure of c, it is also in C

```lean
lemma constructible_of_finite_union' {n : ℕ} (t : Fin n → LocCloseds X) :
    is_constructible (⋃ (i : Fin n), t i : Set X) := by
    induction n with
    | zero =>
        apply is_constructible.op
        simp
    | succ n ih =>
        let t' : Fin n → LocCloseds X := λ ⟨a, ha⟩ => t ⟨a, Nat.lt_succ_of_lt ha⟩
        have H : (⋃ (i : Fin (n+1)), t i : Set X) =
            (⋃ (i : Fin n), t' i : Set X) ∪ (t ⟨n, Nat.lt_succ_self _⟩) := by
            . ext x
              constructor
              . intro hx
                rw [Set.mem_iUnion] at hx
                rcases hx with ⟨⟨i, hi'⟩, hi⟩
                by_cases (i < n)
                . left
                  rw [Set.mem_iUnion]
                  use ⟨i, by assumption⟩
                . have hin : i = n := by
                    apply Nat.le_antisymm
                    . apply Nat.le_of_lt_succ
                      assumption
                    . rw [←Nat.not_lt]
                      assumption
                  right
                  convert hi
                  symm
                  assumption
              . intro hx
                rw [Set.mem_iUnion]
                rcases hx with hx | hx
                . rw [Set.mem_iUnion] at hx
                  rcases hx with ⟨⟨i, hi'⟩, hi⟩
                  use i
                  convert hi
                  simp
                  apply Nat.lt_succ_of_lt
                  assumption
                . use n
                  convert hx
                  simp
        rw [H]
        apply is_constructible_un
        . apply ih
        . apply is_constructible_loc_closed'
```